THE CHINESE UNIVERSITY OF HONG KONG
MATH3270B
HOMEWORK3 SOLUTION

Question 1:

(1) The characteristic equation is $r^3 + y = 0$, we deduce that the roots are $r = 0, i, -i$. Hence the general solution of homogeneous equation is $y = C_1 + C_2 \sin t + C_3 \cos t$. Next we want to find a particular solution of inhomogeneous equation. First, we have:

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & \sin t & \cos t \\ 0 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = -1.$$

Then,

$$W_1(y_1, y_2, y_3) = \begin{vmatrix} 0 & \sin t & \cos t \\ 0 & \cos t & -\sin t \\ 1 & -\sin t & -\cos t \end{vmatrix} = -1.$$

Similarly, $W_2(y_1.y_2, y_3) = \sin t$, $W_3(y_1.y_2, y_3) = \cos t$. By the formula of variation of parameters, we deduce a particular solution

$$y^* = \sum_{i=1}^{3} y_i \int \frac{g(s)W_i}{W(s)} ds = \ln(\sec t + \tan t) - t \cos t + (\sin t) \ln \cos t.$$

By the initial data, we deduce that $C_1 = 0, C_2 = 1, C_3 = 2$, and the solution to the equation is

$$y = \sin t + 2 \cos t + \ln(\sec t + \tan t) - t \cos t + (\sin t) \ln \cos t.$$

(2) The characteristic equation is $(r^2 + 1)^2 = 0$, we deduce that the roots are $r = i$(double), $r = -i$(double). Hence the general solution of homogeneous equation is $y = (C_1+C_2 t) \sin t + (C_3+C_4 t) \cos t$. We use the method of undetermined coefficients to find a particular solution. Since $i, -i$ are double roots of the characteristic equation, we assume the particular solution has the form $y^* = t^2(c_1 \sin t + c_2 \cos t)$, we calculate the second and fourth derivative and deduce that:

$$\begin{aligned} y &= t^2(c_1 \sin t + c_2 \cos t), \\ y'' &= 2(c_1 \sin t + c_2 \cos t) + 4t(c_1 \cos t - c_2 \sin t) + t^2(-c_1 \sin t - c_2 \cos t), \\ y^{(4)} &= 12(-c_1 \sin t - c_2 \cos t) + 8t(-c_1 \cos t + c_2 \sin t) + t^2(c_1 \sin t + c_2 \cos t). \end{aligned} \tag{1}$$

We substitute above to the equation and deduce that $y^* = -\frac{1}{8} t^2 \cos t$. Then using the initial data we determined the solution of the equation is

$$y = (\frac{1}{2} + \frac{5}{8}t)\sin t + (2 - \frac{1}{2}t)\cos t - \frac{1}{8}t^2\cos t.$$

(3) The characteristic equation is $r^3 - r^2 + r - 1 = (r-1)(r^2+1) = 0$, we deduce the roots are $r = 1, i, -i$. Hence the general solution of homogeneous equation is $y = C_1 e^t + C_2 \sin t + C_3 \cos t$. We use the same way as in (1). We have:

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^t & \sin t & \cos t \\ e^t & \cos t & -\sin t \\ e^t & -\sin t & -\cos t \end{vmatrix} = -2e^t.$$

And then $W_1 = -1$, $W_2 = e^t(\sin t + \cos t)$, $W_3 = e^t(\cos t - \sin t)$. Hence we deduce that

$$y^* = -\frac{1}{2}\cos t \ln(\cos t) + \frac{1}{2}\sin t \ln(\cos t) - \frac{1}{2}(t\cos t + t\sin t) + \frac{1}{2}e^t \int_0^t \frac{e^s}{\cos s} ds.$$

Use the initial data, we have $C_1 = \frac{3}{2}$, $C_2 = -\frac{5}{2}$, $C_3 = \frac{1}{2}$. Hence, the solution is:

$$y = \frac{3}{2}e^t - \frac{5}{2}\sin t + \frac{1}{2}\cos t + y^*.$$

(4) The characteristic equation is $r^4 - r^3 - r^2 + r = r(r-1)^2(r+1) = 0$, we deduce that roots are $r = 1$(double),$r = 0, -1$. Hence the general solution of homogeneous equation is $y = C_1 + (C_2 + C_3 t)e^t + C_4 e^{-t}$. Use the same way as in (2), we assume the particular solution has the form

$$y^* = t(a_2 t^2 + a_1 t + a_0) + (b_1 t + b_0)\sin t + (c_1 t + c_0)\cos t,$$

For the polynomial part, it's easy to deduce that $a_2 = \frac{1}{3}$, $a_1 = 1$, $a_0 = 12$. For the trigonometric function part, we calculate the each order derivative and deduce that:

$$\begin{aligned} y' &= (b_1 - (c_0 + c_1 t))\sin t + (c_1 + (b_0 + b_1 t))\cos t, \\ y'' &= -(2c_1 + b_0 + b_1 t)\sin t + (2b_1 - (c_0 + c_1 t))\cos t, \\ y''' &= (-3b_1 + c_0 + c_1 t)\sin t + -(3c_1 + b_0 + b_1 t)\cos t, \\ y'''' &= (4c_1 + b_0 + b_1 t)\sin t + (-4b_1 + c_0 + c_1 t)\cos t. \end{aligned} \tag{2}$$

We substitute (2) to the equation and finally deduce the particular solution is

$$y^* = \frac{t^3}{3} + t^2 + 12t + (\frac{3}{4} + \frac{1}{4}t)\sin t + (\frac{1}{2} - \frac{1}{4}t)\cos t.$$

Then we use the initial data to find that $C_1 = 12$, $C_2 = -\frac{113}{8}$, $C_3 = \frac{21}{4}$, $C_4 = \frac{21}{8}$. Hence, the solution to the IVP is

$$y = 12 - \frac{113}{8}e^t + \frac{21}{4}te^t + \frac{21}{8}e^{-t} + y^*.$$

Question 2: Firstly, the characteristic equation of the homogeneous equation is $r^3-3r^2+3r-1 = (r-1)^3 = 0$, hence the general solution of the homogeneous equation is $y = (C_1 + C_2 t + C_3 t^2)e^t$. Next, by the variation of parameters, we have

$$W(y_1, y_2, y_3) = e^t \begin{vmatrix} 1 & t & t^2 \\ 1 & t+1 & t^2+2t \\ 1 & t+2 & t^2+4t+2 \end{vmatrix} = 2e^t.$$

Similarly, we have $W_1 = t^2 e^t$, $W_2 = -2te^t$, $W_3 = e^t$. Finally, we have

$$\begin{aligned} y^* &= \sum_{i=1}^{3} y_i \int \frac{g(s)W_i}{W(s)} ds \\ &= \frac{1}{2}\int_{t_0}^{t} e^{(t-s)}(t-s)^2 g(s) ds \\ &= \frac{1}{2}e^t \int_{t_0}^{t} (\frac{t}{s} - 1)^2 ds. \end{aligned} \tag{3}$$

And we deduce that $Y(t) = -te^t \ln|t|$.

Question 3:

(1) By the formula, we deduce that

$$\begin{aligned} y' &= \varphi_1' v + v'\varphi_1, \\ y'' &= \varphi_1'' v + 2\varphi_1' v' + v''\varphi_1, \\ y''' &= \varphi_1''' v + 3\varphi_1'' v' + 3\varphi_1' v'' + v'''\varphi_1. \end{aligned} \tag{4}$$

We substitute above to the equation, and deduce that

$$\varphi_1 v''' + (3\varphi_1' + p_2\varphi_1)v'' + (3\varphi_1'' + 2p_2\varphi_1' + p_1\varphi_1)v' = 0, \tag{5}$$

that is,

$$\varphi_1 w'' + (3\varphi_1' + p_2\varphi_1)v' + (3\varphi_1'' + 2p_2\varphi_1' + p_1\varphi_1)w = 0.$$

(2) By (1), we deduce that w satisfies the following equation, since in this question $\varphi_1 = \varphi_1' = \varphi_1'' = \varphi_1^{(3)}$,:

$$(2-t)w'' = (t-3)w', \tag{6}$$

we solve this equation and deduce that $w = C_1(1-t)e^{-t}+C_2$, and hence $v = C_1te^{-t}+C_2t+C_3$. Therefore, we deduce that the general solution is $y = C_1t + C_2te^t + C_3e^t$.

Question 4:

Proof 1 Use the hint given in the question. Since the coefficients $p_i(t)$ are continuous on the interval, we deduce that the equation will have a fundamental set of solutions which denoted to be $\{y_1, y_2...y_n\}$. Since $\{y_1...y_n\}$ are linearly independent, we deduce that they span a n-dim linear space which denoted to be H. Now since $\{\varphi_1...\varphi_n\}$ belongs to H and are also linearly independent, we have the set $\{\varphi_i\}_{i=1}^n$ can also span the space H. That is, y_i can be represented as a linear combination of $\{\varphi_i\}_{i=1}^n$. Therefore, we deduce that each solution of this differential equation can be written as a linear combination of $\{\varphi_i\}_{i=1}^n$, which means $\{\varphi_i\}_{i=1}^n$ can be regarded as a fundamental set of solution, and hence their Wronskian is of course nonzero.

Proof 2 We give another proof. Suppose there exists constants c_i s.t.

$$c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n = 0,$$

now, since the differential operator is linear, we deduce that we have for $k \le n-1$,

$$c_1\varphi_1^{(k)} + \cdots + c_n\varphi_n^{(k)} = 0.$$

Now consider the following system of linear equations:

$$\begin{cases} c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n = 0, \\ c_1\varphi_1' + c_2\varphi_2' + \cdots + c_n\varphi_n' = 0, \\ \cdots \\ c_1\varphi_1^{(n-1)} + c_2\varphi_2^{(n-1)} + \cdots + c_n\varphi_n^{(n-1)} = 0, \end{cases} \tag{7}$$

we consider c_i to be known. Now since $\{\varphi_1...\varphi_n\}$ is linearly independent, we must have $c_1 = 0$, i.e. the (7) will have only zero solution. Hence the determinant of coefficients is nonzero, that is $W(\varphi_1...\varphi_n) \neq 0$.

Question 5: The idea is that construct a differential equation such that the fundamental set of that equation contain the set given in the question.
The term $e^{r_1}t$ is corresponding to the root $k_1 = r_1$, and the terms $e^{r_2}t,\ e^{r_2}t^2$ are corresponding to the (exactly the triple roots because we should add the constant term to it) roots $k_2 = k_3 = k_4 = 0$, and the terms $e^{at}\sin t,\ e^{at}\cos t$ are corresponding to the roots $k_5 = a + bi,\ k_6 = a - bi$. Hence, the characteristic equation is

$$(k - r_1)k^3(k - (a + bi))(k - (a - bi)) = 0.$$

Hence, the corresponding differential equation is

$$y^{(6)} - (2a + r_1)y^{(5)} + (a^2 + b^2 - 2ar_1)y^{(4)} - r_1(a^2 + b^2)y^{(3)} = 0. \tag{8}$$

And the fundamental set of that equation is

$$\{C_1 e^{r_1 t},\ C_2,\ C_3 t,\ C_4 t^2,\ e^{at}(C_5 \sin bt + C_6 \cos bt)\},$$

the constants $C_i \neq 0$, and the fundamental set is of course linearly independent. Therefore, the set given in the question is subset of it which is linearly independent too.